MapReduce Algorithms

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Complexity Model for MapReduce

Minimum Spanning Tree in MapReduce

Computing Dense Subgraph in MapReduce
Complexity Model for MapReduce: $\mathcal{MRC}^i$

- Input: finite sequence of pairs: $(k_i, v_i) : (key, value)$.
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Complexity Model for MapReduce: $\mathcal{MRC}^i$

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- Total Input length $= \sum_i k_i + v_i = n$
- The algorithm executes a sequence of map and reduce tasks $(\mu_1, \rho_1, \mu_2, \rho_2, \ldots, \mu_R, \rho_R)$
Consider an $\epsilon > 0$.

- Each map and reduce task requires $n^{1-\epsilon}$ space
Complexity Model for MapReduce: $MRC^i$

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- The number of rounds $R = O((\log n)^i)$
Complexity Model for MapReduce

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▶ Therefore $MRC^0$ is the class of problems that requires only constant number of rounds with sublinear amount of memory in each machine, and sublinear number of machines altogether.

▶ There are other classes defined such as $MR$ model, where more explicit time+communication complexity of a problem is accounted for.
Given a graph $G = (V, E)$ on $|V| = N$ vertices and $|E| = M \geq N^{1+c}$ edges for some constant $c > 0$ ($n$ still denotes the length of the input and not the number of vertices)

Compute Minimum Spanning Tree of the graph.
Minimum Spanning Tree (MST) in MapReduce

- Fix a number $k$

Randomly partition the set of vertices into $k$ equally sized subsets, $V = V_1 \cup V_2 \cup \ldots \cup V_k$, with $V_i \cap V_j = \emptyset$ for $i \neq j$ and $|V_i| = N/k$ for all $i$. For every pair \{i, j\}, let $E_{i,j} \subseteq E$ be the set of edges induced by the vertex set $V_i \cup V_j$. Denote the resulting subgraph by $G_{i,j} = (V_i \cup V_j, E_{i,j})$. 
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$\text{where } E_{i,j} = \{(u, v) \in E | u, v \in V_i \cup V_j\}$.
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For every pair $\{i, j\}$, let $E_{i,j} \subseteq E$ be the set of edges induced by the vertex set $V_i \cup V_j$.

$$E_{i,j} = \{(u, v) \in E \mid u, v \in V_i \cup V_j\}$$
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- Place each $G_{i,j}$ in a single machine
- Compute MST $M_{i,j}$ of $G_{i,j}$
- Let $H = (V, \bigcup_{i,j} M_{i,j})$
- Compute MST of $H$ in a single machine
Minimum Spanning Tree (MST) in MapReduce

Theorem

The algorithm computes MST correctly.

- If an edge $e$ is discarded, that is $e \in E(G)$ but $e \notin E(H)$: show that $e$ is not part of a MST.
Minimum Spanning Tree (MST) in MapReduce

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- If an edge $e$ is discarded, that is $e \in E(G)$ but $e \notin E(H)$: show that $e$ is not part of a MST.
- Every edge is present in at least one $G_{i,j}$
- If an edge does not appear in $M_{i,j}$, then there exists a cycle in $G_{i,j}$ such that $e$ is the heaviest weight edge in that cycle. This implies $e$ cannot be part of the MST of $G$. 
Lemma

Let \( k = N^{c/2} \) then with high probability the size of every \( E_{i,j} \) is \( \tilde{O}(N^{1+c/2}) \).

- With high probability each part has \( \tilde{O}(N^{1+c/2}) \) edges. Therefore, the total input size to any reducer is \( O(n^{1-\epsilon}) \).
Minimum Spanning Tree (MST) in MapReduce

- There are $N^c$ total parts, each producing a spanning tree with $2N/k - 1 = O(N^{1-c/2})$ edges.
- Thus the size of $H$ is bounded by $\tilde{O}(N^{1+c/2}) = O(n^{1-\epsilon})$, again small enough to fit into the memory of a single machine.
Minimum Spanning Tree (MST) in MapReduce

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\[ |E_{i,j}| \leq \sum_{v \in V_i} \deg(v) + \sum_{v \in V_j} \deg(v). \]
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Let $W_i = \{v \in V : 2^{i-1} < \deg(v) \leq 2^i\}$. Hence $W_1$ is the set of vertices with degree at most 2, $W_2$ is the set of vertices with degrees 3 and 4, and so on.
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- There are $\log N$ total groups.
Lemma
Let $k = N^{c/2}$ then with high probability the size of every $E_{i,j}$ is $	ilde{O}(N^{1+c/2})$.

- How many vertices from $W_i$ are mapped to $V_j$?
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- How many vertices from $W_i$ are mapped to $V_j$?
- If $|W_i| < 2N^{c/2} \log N$ then
  \[ \sum_{v \in W_i} \deg(v) \leq 2N^{1+c/2} \log N = \tilde{O}(N^{1+c/2}). \]
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  $\sum_{v \in W_i} \deg(v) \leq 2N^{1+c/2} \log N = \tilde{O}(N^{1+c/2})$.
- If the group is large, using concentration inequality, we can show the number of vertices mapped from any particular group to $V_j$ is small.
- Overall, the total degree in any part remains bounded by $\tilde{O}(N^{1+c/2})$.
Computing Dense Subgraph in MapReduce

Given an undirected graph \( G = (V, E) \), compute a subset of nodes \( S \subseteq V \) such that \( \frac{|E(S)|}{|V(S)|} \) is maximized.

- Community Mining
- Computational Biology
- Link Spam Detection
- Efficient Indexing for Reachability Queries
We will show an algorithm in that computes the dense subgraph in multiple passes, but in a single machine, and use $O(n \log n)$ memory at any pass.
Computing Dense Subgraph in Streaming Setting

- We will show an algorithm in that computes the dense subgraph in multiple passes, but in a single machine, and use $O(n \log n)$ memory at any pass.
- Exercise convert the algorithm into MapReduce framework.
Let $\epsilon > 0$ be a parameter.

We start with the given graph $G$, compute the current density $\rho(G)$ and remove all nodes whose degree is less than $(2 + 2\epsilon)\rho(G)$.

If the remaining graph is nonempty, recurse on the remaining graph.

Return the graph from the round which has highest density.
Computing Dense Subgraph in Streaming Setting

Lemma

Algorithm obtains $(2 + 2\epsilon)$-approximation to the densest subgraph problem.

- Consider the round in which a vertex from the optimum subgraph $S^*$ is removed for the first time.
- Consider a $i \in S^*$ that is removed.
- We have

$$\rho(S^*) \leq \text{deg}_{S^*}(i) \leq \text{deg}_S(i) \leq (2 + 2\epsilon)\rho(S)$$
Lemma

Algorithm terminates in $\log_{1+\epsilon}(n)$ rounds, $n = |V|$.

- Exercise: show that after each round, the number of vertices reduce by a factor of $\frac{1}{(1+\epsilon)}$.
- Exercise: show how to convert the algorithm into a $MRC^1$ algorithm.